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Letter to the Editor

## Some remarks on convergence of curvelet transform of piecewise smooth functions ☆

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### ABSTRACT

The convergence of the curvelet transform and the best  $M$ -term non-linear curvelet approximation of a piecewise smooth function is studied. Theorems that relate convergence rates to the smoothness of the function are given. Although the given smoothness conditions for convergence are not sharp, the theorems demonstrate that convergence rates increase when smoothness increases. The main result is that the log-factor in the well-known approximation rate  $M^{-2}(\log(M))^3$  is unnecessary if the function and edges that separate smooth regions have more smoothness than in traditional assumptions.

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## 1. Introduction

In applications such as image compression, de-noising and feature extraction, some transform (e.g. the Fourier, wavelet or curvelet transform) is usually first applied. The final results are often strongly dependent on the convergence properties of the transform that was used.

When the curvelet transform was introduced, its most striking feature was that while being a non-adaptive transform, it still provided an almost optimal approximation rate for cartoon type images. To be a bit more specific, if the function  $f \in L_2(\mathbb{R}^2)$  has a discontinuity on the twice differentiable curve and otherwise  $f$  is twice differentiable, with bounded derivatives, then the  $M$ -term non-linear approximation error with curvelets is bounded by  $M^{-2} \log(M)^3$  [1]. This same approximation rate has been proved also for contourlets and shearlets [2,3]. Similar optimality results have been presented also for inverse problems [4].

What, however, if the smoothness increases? Then some adaptive transforms such as the bandelet-transform remain optimal, but the curvelet transform is not close to optimal anymore [5]. However, smoothness of function has some connections to values of transforms that are based on parabolic scaling [6–8]. This article presents theorems stating that even if optimality is lost, curvelets still gain some advantage from the increasing smoothness. These results are refined versions from [9].

The article is organized as follows: First, curvelets and the class of functions that we are studying are defined. The results are given then in a separate section.

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## 2. Mathematical preliminaries and definitions

For simplicity, throughout the article we assume that all functions are real-valued. Usual norm in  $L_2(\mathbb{R}^2)$  is used, i.e.  $\|f\|_{L_2(\mathbb{R}^2)}^2 := \int_{\mathbb{R}^2} (f(x))^2 dx$ . Diagonal matrix performing parabolic scaling is  $D_a = \text{diag}(a, \sqrt{a})$  and  $R_\theta$  is a matrix affecting the planar rotation of  $\theta$  radians in the counter-clockwise direction. Space  $C^n$  is the space of  $n$  times continuously differentiable functions. By  $f_{M,B}$  we denote best  $M$ -term non-linear approximation of  $f$  in frame  $B$ . A recommendable for the details about frames is for example [10].

### 2.1. Curvelets

There exist different constructions of curvelets. Theorems that we present in Section 3 hold at least for curvelets used in [6,11,1]. Curvelets used in [6,11] (in context of continuous curvelet transform) and curvelets used in [1] (in context of curvelet frames for  $L_2(\mathbb{R}^2)$ ) are actually very similar: all curvelets  $\gamma_{a\theta b}$  at scale  $a$  are generated by translation and rotation of one single function  $\gamma_{a00}$ , i.e.

$$\gamma_{a\theta b}(x) = \gamma_{a00}(R_{-\theta}(x - b)) \quad \text{for } x \in \mathbb{R}^2. \quad (1)$$

When designing  $\gamma_{a00}$ , one essential property is that most of its energy should be concentrated on rectangle with width about  $a$  and length about  $a^{1/2}$  and orientation of rectangle is same for all scales  $a$ . Let now line  $l$  be parallel to side of this rectangle that had length about  $a^{1/2}$  and  $l$  is going through origin. We call  $\{x \in \mathbb{R}^2: x = R_\theta y + b, y \in l\}$  as major axis of  $\gamma_{a\theta b}$ .

More details about construction and properties of  $\gamma_{a00}$  and range for parameters  $a$ ,  $b$  and  $\theta$  can be found from [6,11,1].

### 2.2. Functions that are smooth apart from smooth curves

We define here the class of functions that are considered in later lemmas and theorems. Smooth curves in  $\mathbb{R}^2$  are defined first.

**Definition 2.1.** Assume that a plane curve  $S \in \mathbb{R}^2$  has a tangent in all points and  $s_{p,r}$  is a square centered at  $p$  and having the side length  $r$ . Inside this square, the coordinates are defined so that the  $x_1$ -axis is parallel to the tangent of  $S$  at  $p$ , the  $x_2$ -axis is orthogonal to the  $x_1$ -axis and the origin is in  $p$ . If there exists  $r_0 > 0$  s.t.  $\forall r < r_0$  inside  $s_{p,r}$ , the curve  $S$  is the function  $g_{p,r}: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g_{p,r} \in C^n$ , and all first  $n$  derivatives of  $g_{p,r}$  are bounded by the constant  $C$  (independent of  $p$  and  $r$ ), then we say that  $S$  is  $C^n$  smooth with bounded derivatives.

Function  $\chi_A: \mathbb{R}^2 \rightarrow \{0, 1\}$  of a set  $A$  is defined as

$$\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \quad (2)$$

Next we define the class  $F_{N,n}$  of functions that are essentially  $N$  times differentiable apart from curves that are  $n$  times differentiable.

**Definition 2.2.** We say that  $f \in F_{N,n}$  if it is of the form  $f = \chi_{\mathbb{R}^2 \setminus A} f_1 + \chi_A f_2$ , where  $f_1, f_2 \in C^N(\mathbb{R}^2)$  and have bounded derivatives and compact supports, and  $A$  is a compact set with the boundary curve  $S$  that is  $C^n$  smooth with bounded derivatives.

We restrict here to smoothness indexes  $N$  and  $n$  that are integers.

## 3. Results

The following theorem is the main result.

**Theorem 3.1.** Let  $B$  be a curvelet tight frame for  $L_2(\mathbb{R}^2)$  (details in [1]). If  $f \in F_{3,3}$ , then  $\|f - f_{M,B}\|_{L_2(\mathbb{R}^2)}^2 \lesssim M^{-2}$ .

If  $f \in F_{2,2}$ , the well-known bound is  $M^{-2}(\log(M))^3$  [1]. Therefore higher, but still finite, smoothness for  $f$  allows us to remove the factor  $(\log(M))^3$  from the estimate of the best  $M$ -term non-linear approximation error.

The following lemma is an essential ingredient in the proof of Theorem 3.1. We present it here because in some cases it might be interesting on its own. It essentially shows that by assuming more smoothness for  $f$  and  $S$ , the transform will decay faster when the orientation turns away from the orientation of  $S$ .

**Lemma 3.2.** Let  $f \in F_{N,n}$ ,  $N \geq n$ ,  $p \in S$  be the point that minimizes  $L = |D_{1/a} R_{-\theta}(b - p)|$  and  $\theta' \approx ka^{1/2}$  be the angle between the major axis of  $\gamma_{a\theta b}$  and the tangent of  $S$  at point  $p$ . Then for any fixed  $K > 0$  and  $0 < \varepsilon < 2$

$$\left| \int_{\mathbb{R}^2} f(x) \gamma_{a\theta b}(x) dx \right| \lesssim \begin{cases} \max\{a^{3/4+N}, \frac{a^{3/4}}{1+L^K}\}, & |\theta'| \lesssim a^{1/2}, \\ \max\{a^{3/4+N}, \frac{a^{3/4}}{1+|k|^K L^K}\}, & n \geq 3, |\theta'| \gtrsim a^{1/2}, |k|^{1-\varepsilon/2} \gtrsim L, \\ \max\{a^{3/4+N}, \frac{a^{3/4}}{1+|k|^{3+\varepsilon} L^K}\}, & n \geq 3, |\theta'| \gtrsim a^{1/2}, |k|^{1-\varepsilon/2} \lesssim L. \end{cases} \quad (3)$$

A similar decay rate in the case  $n = 2$  was presented for contourlets in [2], but the technique used for proof does not directly give any advantage from higher smoothness assumptions.

Proofs for Theorem 3.1 and Lemma 3.2 are available in [12].

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